SS3011 Space Technology and Applications

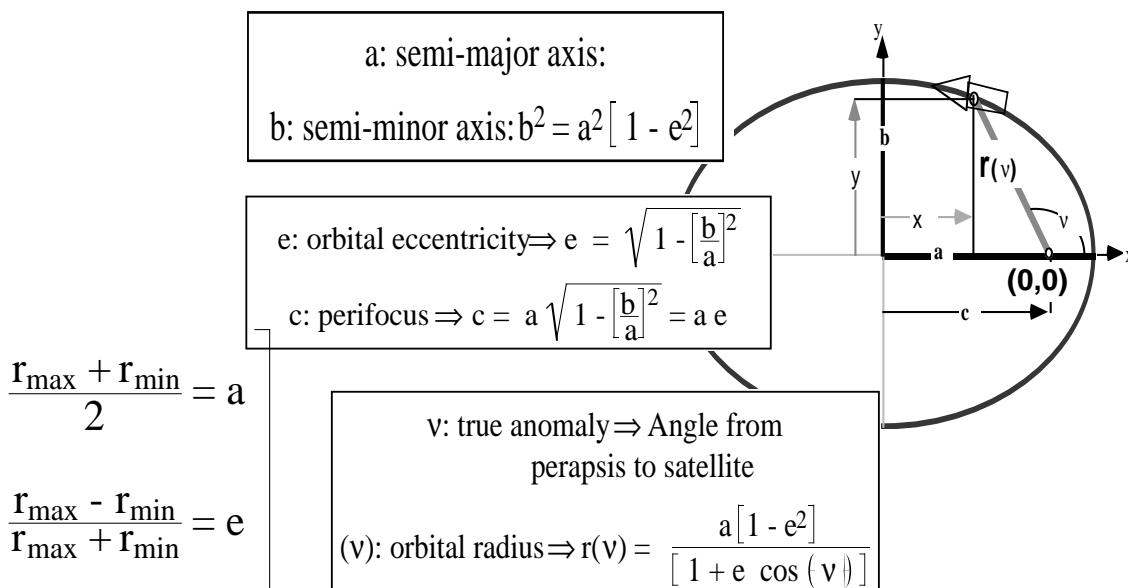## "Orbitology" (cont'd)

N a v a l   P o s t g r a d u a t e   S c h o o l
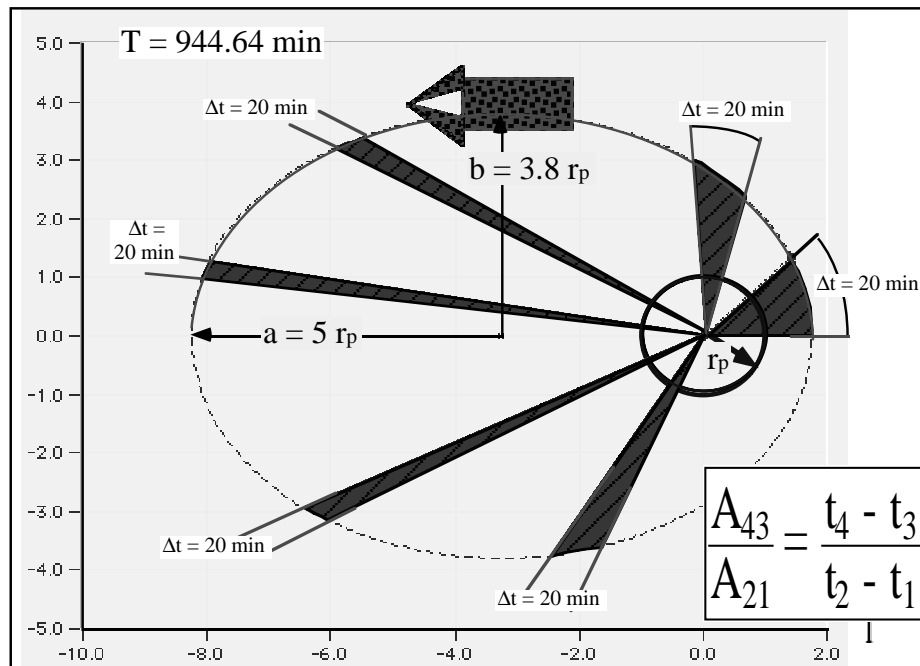M o n t e r e y ,   C a l i f o r n i a

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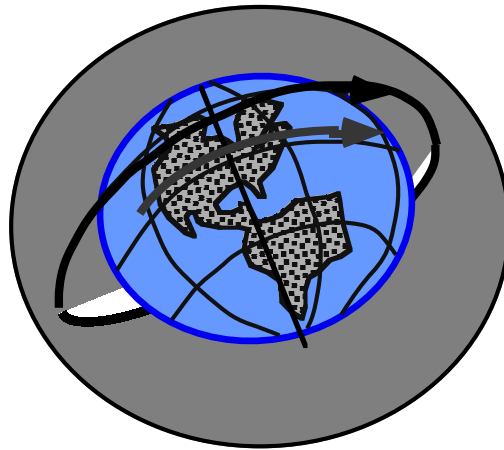
Kepler's First Law: *In a two body universe, the orbit of a satellite around the Earth is an ellipse with the Earth centered at one of the focii*
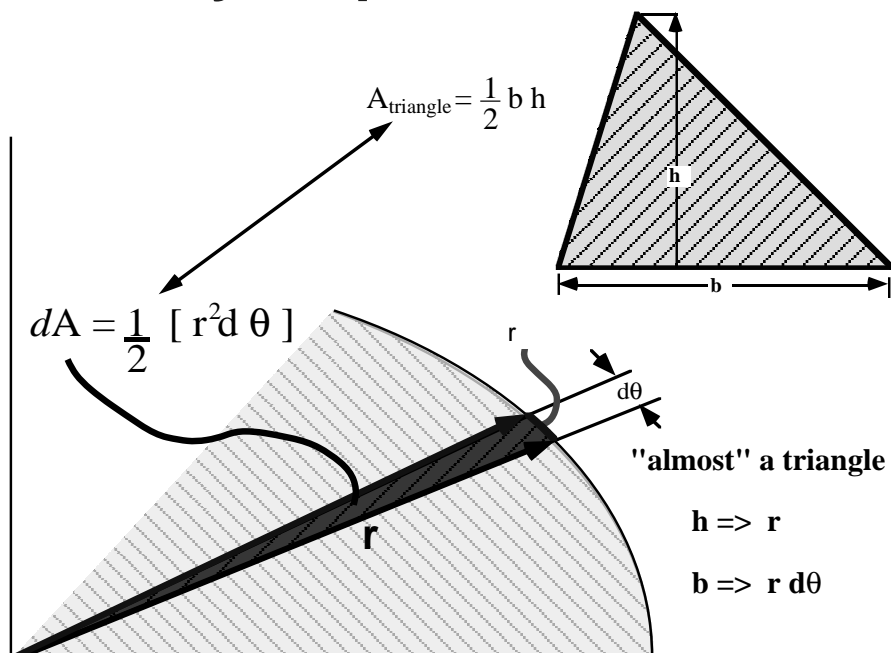
a: semi-major axis:

b: semi-minor axis: $b^2 = a^2 [1 - e^2]$

e: orbital eccentricity $\Rightarrow e = \sqrt{1 - \left[\frac{b}{a}\right]^2}$

c: perifocus $\Rightarrow c = a \sqrt{1 - \left[\frac{b}{a}\right]^2} = a\, e$

$$\frac{r_{max} + r_{min}}{2} = a$$

$$\frac{r_{max} - r_{min}}{r_{max} + r_{min}} = e$$

ν: true anomaly $\Rightarrow$ Angle from perapsis to satellite

(ν): orbital radius $\Rightarrow r(\nu) = \dfrac{a[1 - e^2]}{[1 + e\,\cos(\nu)]}$

N a v a l   P o s t g r a d u a t e   S c h o o l
M o n t e r e y ,   C a l i f o r n i a

Kepler's Second Law

Kepler's Second Law: *In a two body universe, radius vector from the sun (Earth) to the planet (satellite) sweeps out equal areas in equal times*

T = 944.64 min

Δt = 20 min

Δt = 20 min

b = 3.8 r_p

Δt = 20 min

Δt = 20 min

a = 5 r_p

r_p

Δt = 20 min

Δt = 20 min

$$\frac{A_{43}}{A_{21}} = \frac{t_4 - t_3}{t_2 - t_1}$$

Incremental Area Swept Out by Elliptical Arc

$A_{triangle} = \frac{1}{2} b h$

h

b

$dA = \frac{1}{2} [r^2 d\theta]$

r

dθ

r

"almost" a triangle

h => r

b => r dθ

Incremental Area Swept out by an arc

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Area Swept out by
an Elliptical Arc (cont'd)

$$A_{\substack{ellip.\\arc}} = \int_{\nu_0}^{\nu_1} \left[\frac{1}{2} r(\nu)^2 \, d\nu \right] =$$

$$A_{\substack{ellip.\\arc}} = \int_{\nu_0}^{\nu_1} \left[\frac{1}{2} \left[\frac{a[1 - e^2]}{[1 + e \cos(\nu)]} \right]^2 d\nu \right] =$$

$$\frac{1}{2} [a[1 - e^2]]^2 \int_{\nu_0}^{\nu_1} \left[\frac{1}{[1 + e \cos(\nu)]^2} \, d\nu \right]$$

"very difficult" integral

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Area Swept out by
an Elliptical Arc (concluded)

$$A_{\substack{ellip.\\arc}} = \int_{\nu_0}^{\nu_1} \left[\frac{1}{2} r(\nu)^2 \, d\nu \right] =$$

Ouch!

$$\frac{1}{2} [a[1 - e^2]]^2 \left[\frac{e\sqrt{e^2-1}\, \text{Sin}(\nu_1) - 2F_1 - 2e\,\text{Cos}(\nu_1)\,F_1}{(e^2-1)^{3/2}[1 + e\text{Cos}(\nu_1)]} - \frac{e\sqrt{e^2-1}\,\text{Sin}(\nu_0) - 2F_0 - 2e\,\text{Cos}(\nu_0)\,F_0}{(e^2-1)^{3/2}[1 + e\text{Cos}(\nu_0)]} \right]$$

$$\Downarrow$$

$$F_1 = \text{Tanh}^{-1}\left[\frac{(e-1)\,\text{Tan}\left[\frac{\nu_1}{2}\right]}{\sqrt{e^2-1}} \right]$$

$$F_0 = \text{Tanh}^{-1}\left[\frac{(e-1)\,\text{Tan}\left[\frac{\nu_0}{2}\right]}{\sqrt{e^2-1}} \right]$$
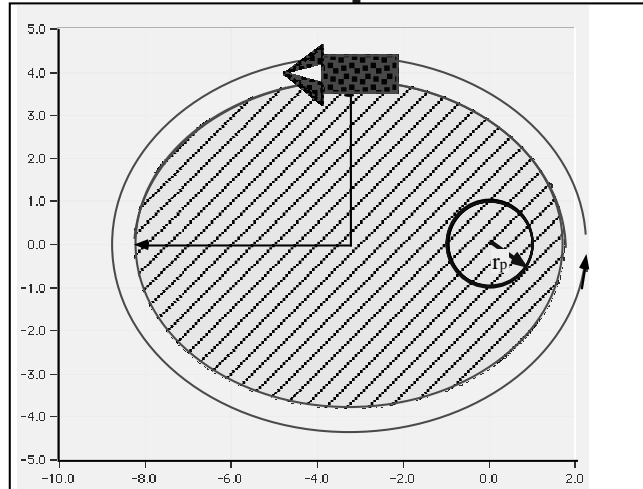
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Total Area of an Elliptical Orbit

But ....we can solve for the total area of an ellipse in *closed form*

$$A_{\substack{ellipse \\ total}} = \int_0^{2\pi} \left[\frac{1}{2} r(\nu)^2 \, d\nu \right] =$$

$$\boxed{a^2 \pi \sqrt{1 - e^2}}$$

$$e = \sqrt{1 - \left[\frac{b}{a}\right]^2} \Rightarrow \sqrt{1 - e^2} = \frac{b}{a} \Rightarrow$$

$$A_{\substack{ellipse \\ total}} = a^2 \pi \sqrt{1 - e^2} = \pi a^2 \frac{b}{a} = \boxed{\pi a b}$$

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Review: Using Kepler's Second Law to Determine Orbital Position

$$\frac{A_{\nu_1 - \nu_0}}{A_{ellipse}} = \frac{t_1 - t_0}{T} \Rightarrow A_{\nu_1 - \nu_0} = \left[a^2 \pi \sqrt{1 - e^2} \right] \frac{t_1 - t_0}{T}$$

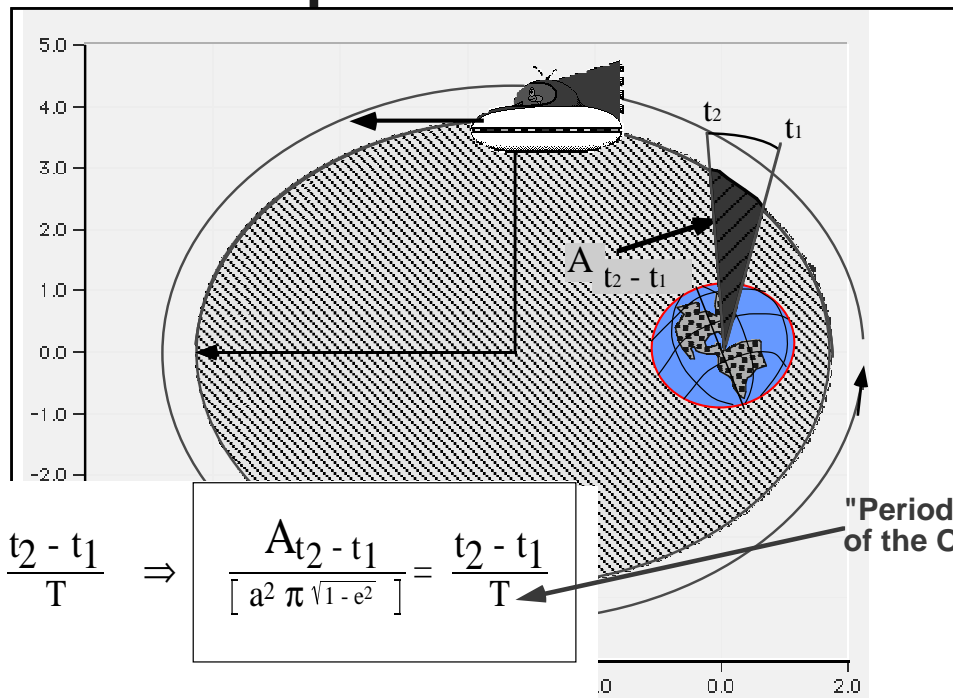$$A_{\nu_1 - \nu_0} = A_{\nu_1 - 0} - A_{\nu_o - 0}$$

$$\Downarrow$$

$$\boxed{\frac{A_{\nu_1}}{a^2} = \frac{A_{\nu_o}}{a^2} + \left[\pi \sqrt{1 - e^2} \right] \frac{t_1 - t_0}{T}}$$

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Mathematical Representation of Kepler's Second Law

$A_{t_2 - t_1}$

t_2 t_1

"Period" of the Orbit

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$$\frac{A_{t_2 - t_1}}{A_{ellipse\ total}} = \frac{t_2 - t_1}{T} \quad \Rightarrow \quad \frac{A_{t_2 - t_1}}{\left[\ a^2\ \pi\ \sqrt{1-e^2}\ \right]} = \frac{t_2 - t_1}{T}$$

$$A_{t_2 - t_1} = \left[\ a^2\ \pi\ \sqrt{1-e^2}\ \right] \times \frac{t_2 - t_1}{T}$$

Review: Graphical Representation of Kepler's Second Law

Graphical Representation of Kepler's Second Law

Or .....

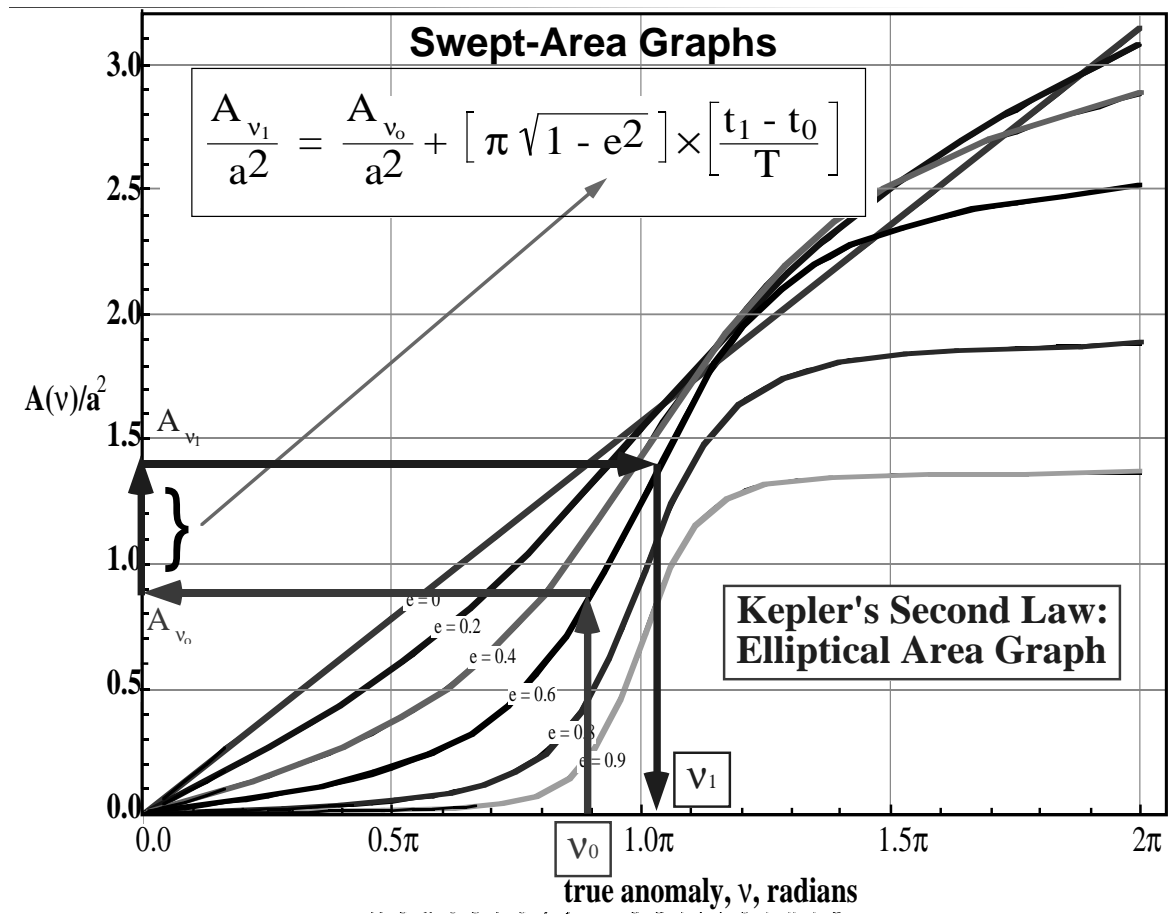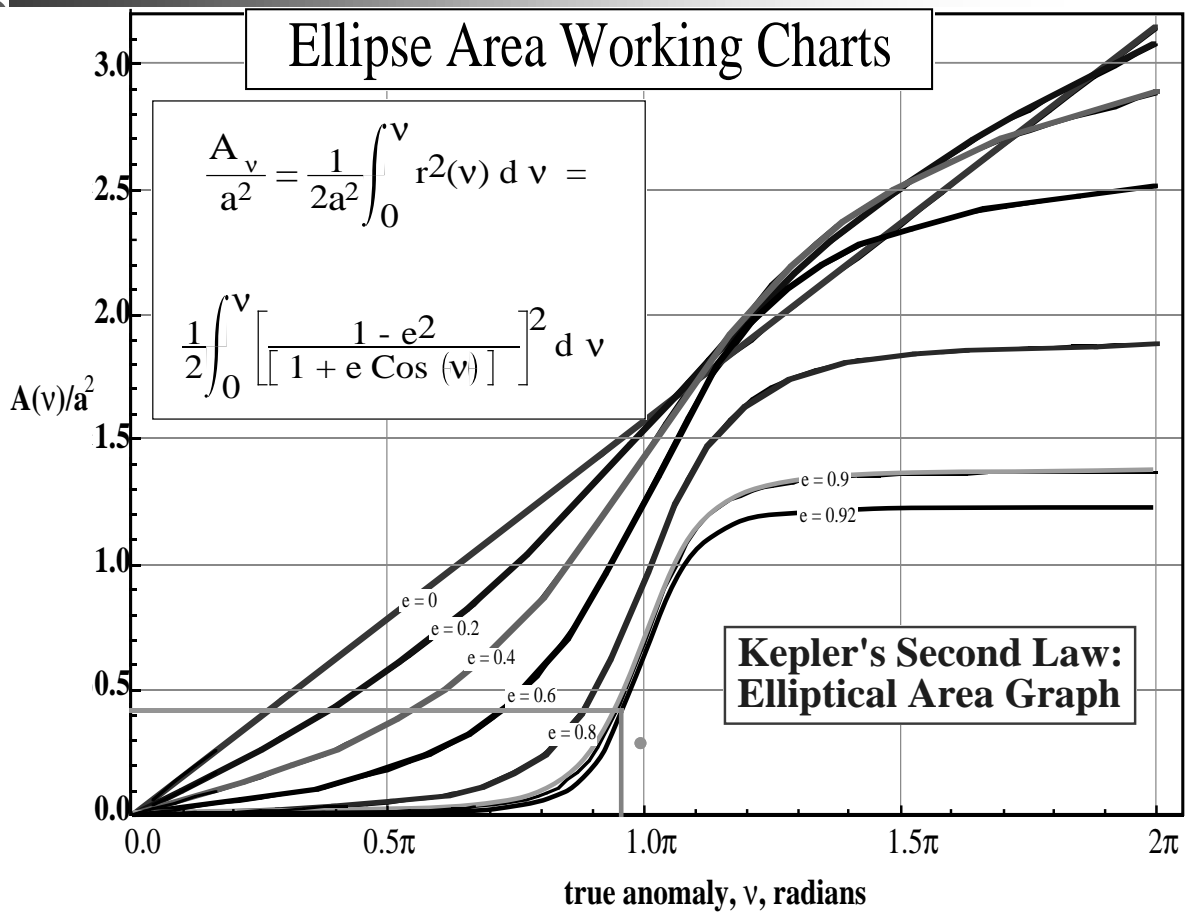

ν_1

t_1 $A_{t_1 - t_0}$ t_0

ν_0
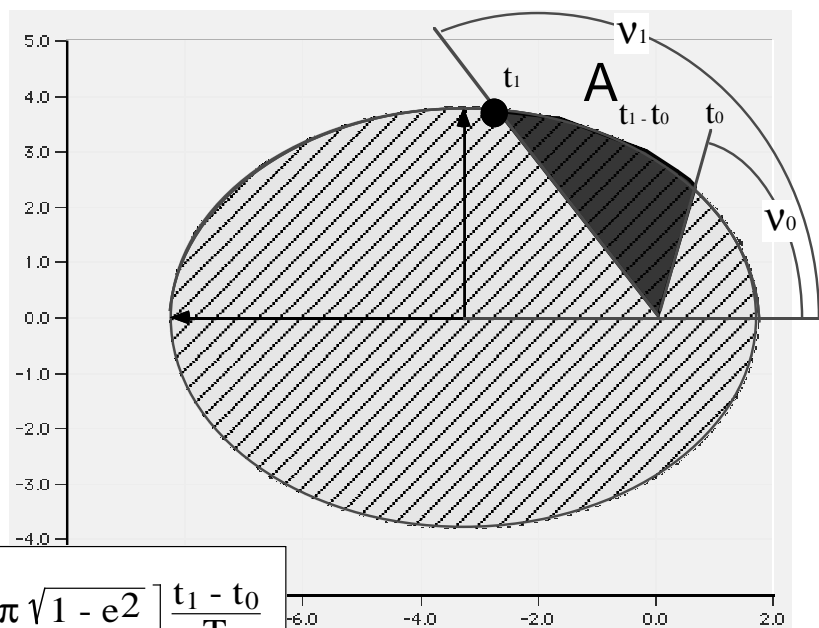
$$\frac{A_{\nu_1}}{a^2} = \frac{A_{\nu_o}}{a^2} + \left[\ \pi\ \sqrt{1-e^2}\ \right]\frac{t_1 - t_0}{T}$$

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Ellipse Area Working Charts

$$\frac{A_\nu}{a^2} = \frac{1}{2a^2} \int_0^\nu r^2(\nu)\, d\nu =$$

$$\frac{1}{2} \int_0^\nu \left[\frac{1 - e^2}{[\, 1 + e \cos(\nu)\,]} \right]^2 d\nu$$

$A(\nu)/a^2$

e = 0.9
e = 0.92
e = 0
e = 0.2
e = 0.4
e = 0.6
e = 0.8

**Kepler's Second Law:
Elliptical Area Graph**

3.0
2.5
2.0
1.5
1.0
0.5
0.0

0.0   0.5π   1.0π   1.5π   2π

true anomaly, ν, radians

Swept-Area Graphs

$$\frac{A_{\nu_1}}{a^2} = \frac{A_{\nu_0}}{a^2} + \left[\, \pi \sqrt{1 - e^2}\,\right] \times \left[\frac{t_1 - t_0}{T} \right]$$

$A(\nu)/a^2$

A_{ν_1}

A_{ν_0}

e = 0
e = 0.2
e = 0.4
e = 0.6
e = 0.8
e = 0.9

ν_1

**Kepler's Second Law:
Elliptical Area Graph**

3.0
2.5
2.0
1.5
1.0
0.5
0.0

0.0   0.5π   1.0π   1.5π   2π

ν_0

true anomaly, ν, radians

Kepler's Third law

Kepler's Third law

• **Kepler's Third Law:** *In a two body universe, square of the period of any object revolving about the sun (Earth) is in the same ratio as the cube of its mean distance*

Kepler's Third Law

Solving for the Orbit Period

$$\text{constant} = \frac{4 a^3 \pi^2}{T^2}$$

$$\mu \equiv \text{constant}$$
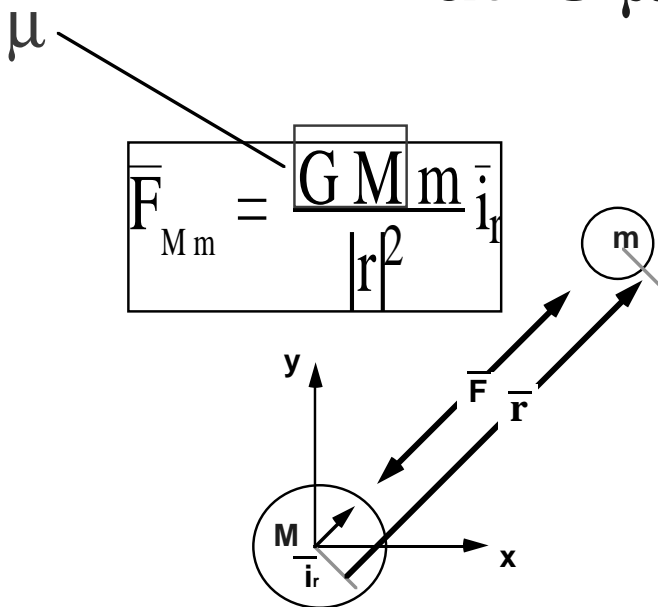
$$T = \frac{2 \pi a^{3/2}}{\sqrt{\mu}}$$

• Third law directly derivable from first and second laws

μ --> **Planetary gravitational parameter**

What is μ?

μ

$$\bar{F}_{Mm} = \frac{G M m}{|r|^2} \bar{i}_r$$

m

y

$\bar{F}$ $\bar{r}$

M
$\bar{i}_r$

x

Isaac Newton, (1642-1727)

Planetary Gravitational Parameter

$$\mu_{earth} = G\, M \approx 6.672 \text{ x } 10^{-11}\, \frac{\text{Nt-m}^2}{\text{kg}^2} \times\ 5.974 \text{ x } 10^{24}\text{kg} =$$

$$3.98565 \text{ x } 10^{14}\, \frac{\text{Nt-m}^2}{\text{kg}} = 3.986 \text{ x } 10^{14}\, \frac{\text{m}^3}{\text{sec}^2} = 1.4076 \text{ x } 10^{16}\, \frac{\text{ft}^3}{\text{sec}^2}$$

$$\mu_{moon} = 4.903 \times 10^3\, \frac{\text{m}^3}{\text{sec}^2}$$

$$\mu_{sun} = 1.327 \times 10^{20}\, \frac{\text{m}^3}{\text{sec}^2}$$

$$\mu_{Mars} = 4.269 \times 10^4\, \frac{\text{m}^3}{\text{sec}^2}$$

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M o n t e r e y ,   C a l i f o r n i a

Elapsed Time Formulae
(Including Kepler's Third Law)

Kepler Second Law

Kepler Third Law

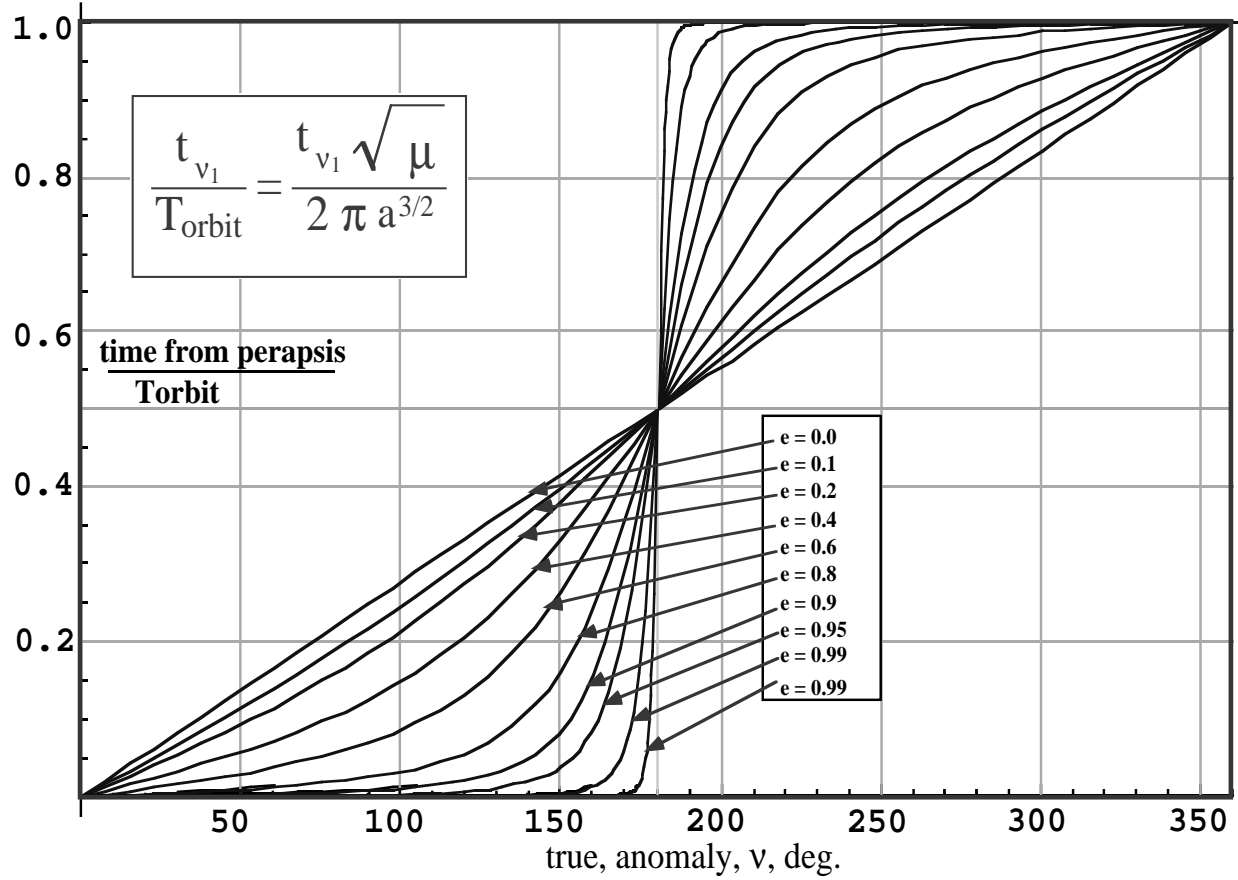$$T\ =\ \frac{2\,\pi\,a^{3/2}}{\sqrt{\mu}}$$

Implicit relationship between *t* and ν

$$\frac{t_{\nu_1}}{T} = \frac{A_{\nu_1}}{A_{ellipse}} =$$

$$\frac{t_{\nu_1}\sqrt{\mu}}{2\,\pi\,a^{3/2}} = \frac{\left[A_{\nu_1}\,/a^2\right]}{\pi\,\sqrt{\left[1 - e^2\right]}} =$$

$$\frac{\dfrac{1}{2}\displaystyle\int_0^{\nu_1}\left[\frac{1 - e^2}{1 + e\cos(\nu)}\right]^2 d\nu}{\pi\,\sqrt{\left[1 - e^2\right]}}$$

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M o n t e r e y ,   C a l i f o r n i a

Kelper's Second Law, Normalized Time vs. true anomaly, elliptical orbit

$$\frac{t_{\nu_1}}{T_{orbit}} = \frac{t_{\nu_1} \sqrt{\mu}}{2 \pi a^{3/2}}$$

$\underline{\text{time from perapsis}}$
   $\overline{\text{Torbit}}$

e = 0.0
e = 0.1
e = 0.2
e = 0.4
e = 0.6
e = 0.8
e = 0.9
e = 0.95
e = 0.99
e = 0.99

true, anomaly, ν, deg.

Time of Flight Graphs (cont'd)

ν_1

t_1

t_0

ν_0

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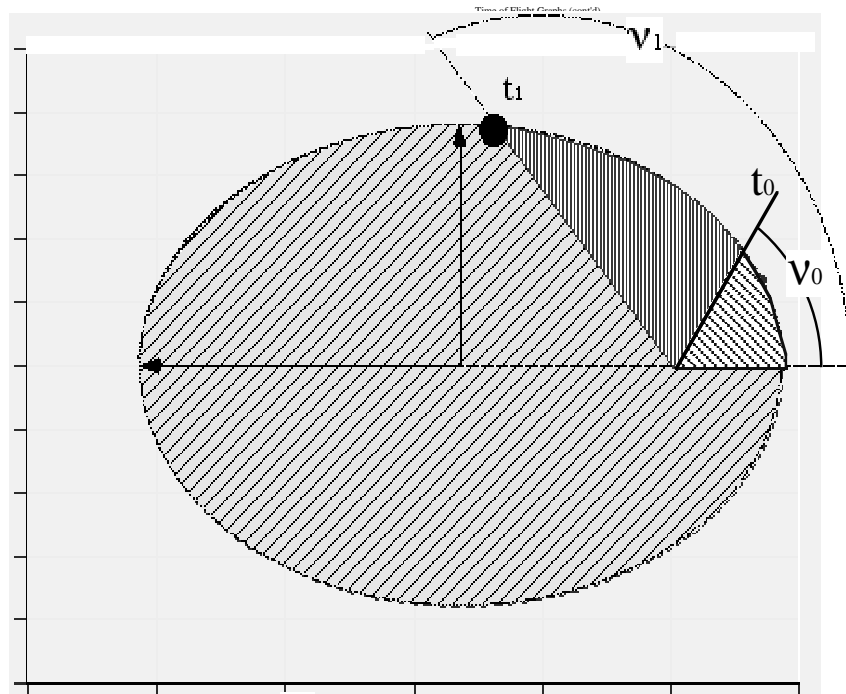
Propogation of Orbital Position

Kelper's Second Law, Normalized Time vs. true anomaly, elliptical orbit

Better form of the "Area Plots"

$\dfrac{\text{time from perapsis}}{\text{Torbit}}$

$\dfrac{t_1 - t_0}{T}$

$\dfrac{t_0}{T}$

e = 0.0
e = 0.1
e = 0.2
e = 0.4
e = 0.6
e = 0.8
e = 0.9
e = 0.95
e = 0.99
e = 0.99

ν_0 ν_1 true, anomaly, ν, deg.

Homework:
Kepler's Second Law

Δt from Pera-Epsilon = **50** hrs **21** min **48** sec **600** msec

0.75 AU   0.25 AU

Epsilon-Hydra 5

Pera-Epsilon

Ap-Epsilon

ν

Potemtkeon

Line-of-apsides

Excelsior
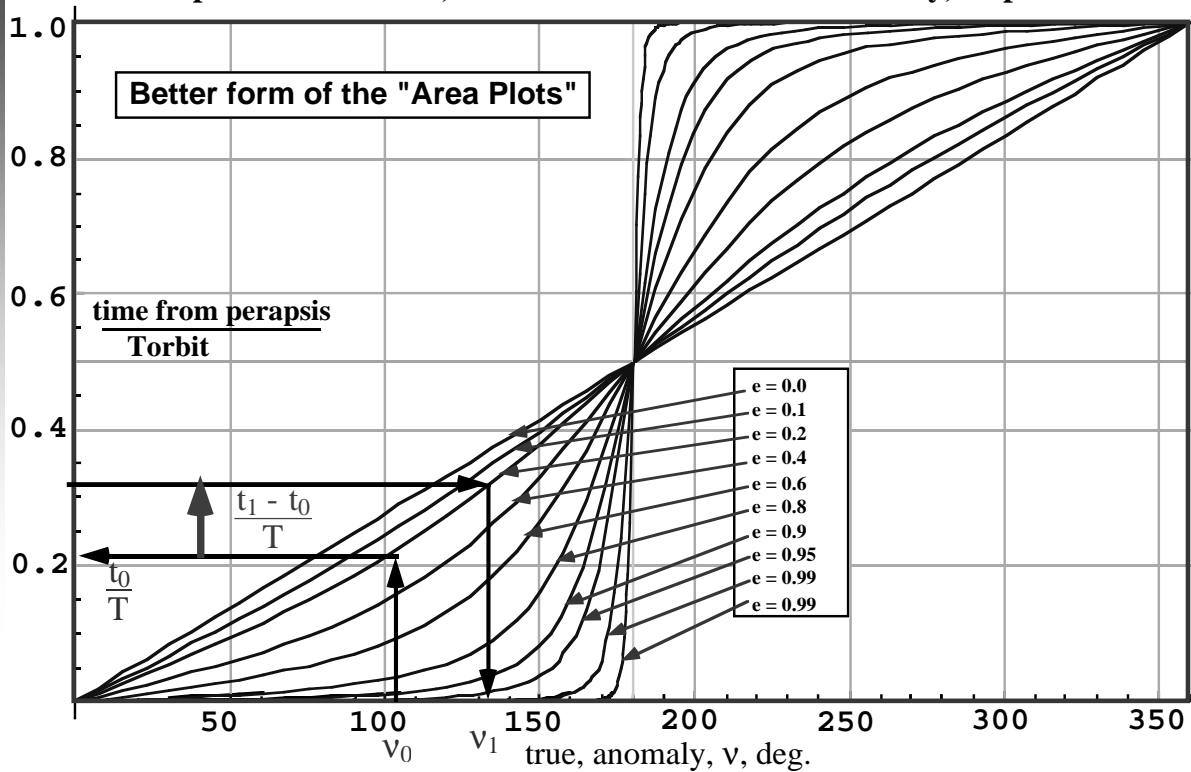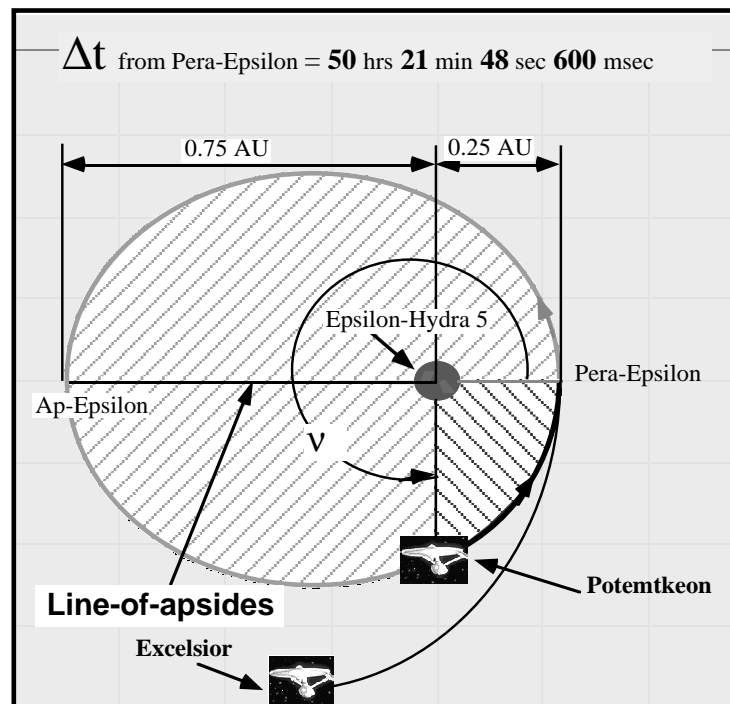
Postscript:Alternate Form of Kepler's Second Law (cont'd)

$$\begin{cases} \text{Let:} \left[t_2 \Rightarrow t_1 \right] \rightarrow t_2 - t_1 = dt \\ \text{Then: } A_{t_2 - t_1} = d\,A(t) \end{cases} \Rightarrow d\,A(t) = \left[a^2\,\pi\,\sqrt{1-e^2} \right] \frac{dt}{T}$$

• **But** $\boxed{d A(t) = \tfrac{1}{2} r^2\, dv}$ *"constant" for a given orbit*

and

$$\boxed{\frac{d\,A(t)}{dt} = \frac{\left[a^2\,\pi\,\sqrt{1-e^2} \right]}{T} = \frac{\tfrac{1}{2} r^2\, dv}{dt} = \tfrac{1}{2} r^2 \frac{dv}{dt}}$$

"Specific" Angular Momentum

N a v a l   P o s t g r a d u a t e   S c h o o l
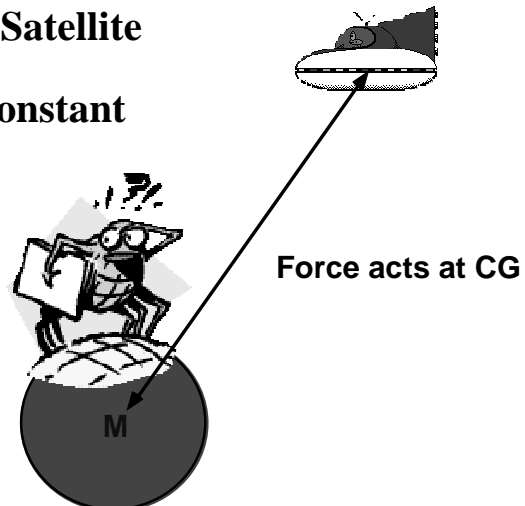M o n t e r e y ,   C a l i f o r n i a

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Kepler's Second Law (Concluded)
Alternate form …

Look at the Problem Physics

• **Gravity Can Exert No Torque on a Satellite**

• **Therefore, Angular momentum is constant**

$$\boxed{r^2 \frac{dv}{dt} = \frac{2\left[a^2\,\pi\,\sqrt{1-e^2} \right]}{T} \equiv l}$$

Force acts at CG

M

N a v a l   P o s t g r a d u a t e   S c h o o l
M o n t e r e y ,   C a l i f o r n i a

Homework:
Kepler's Second Law (cont'd)

• United Federation of Planets Starship *Potemtkeon* is parked in a highly elliptical orbit around the Federation Outpost planet *Epsilon-Hydra 5.*

• *Captain Sulu* commanding *Potemtkeon* has orders to *rendezvous* with Federation Starship *Excelsior* commanded by *Captain Checkov*

• From sensor readings *Captain Sulu* knows that the orbit around *Epsilon-Hydra 5* has a closest approach *(pera-Epsilon)* of *0.25 astronomical units* (AU).* The farthest distance from the planet *(ap-Epsilon)* is *0.75 AU*

*AU ~ 150,000,000 kilometers

•*The Potemtkeon* has passed the orbit *ap-Epsilon* and is headed back towards the closest approach to the planet

• Navigation data shows that the current position of the *Potemtkeon* relative to the planet is EXACTLY PERPENDICULAR to the *Line of Apsides** of the orbit
*(the line connecting the *pera-Epsilon* and the *ap-Epsilon*)

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Homework:
Kepler's Second Law (cont'd)

• The onboard atomic clock shows that it has been exactly *50 hours, 21 minutes, 48 seconds, and 600 miliseconds* (3021.81minutes) since the *Potemtkeon* last passed the *pera-Epsilon* point

• In order to succesfully *rendezvous*, *CaptainCheckov* aboard the *Excelsior* must know the exact orbital eccentricity, e, and the time of arival at the *pera-Epsilon* point from the current position

• What should Captain Sulu tell him?

• If they fail to *rendezvous* at the next encounter, how long will Checkov have to wait until the next *pera-Epsilon* encounter

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Homework:
Kepler's Second Law (concluded)

Homework …

• Hint 1:

$$\left[\begin{array}{l} r_{min} = a(\ 1 - e) \\[2mm] r_{max} = a(\ 1 + e) \end{array}\right]$$

• Hint 2:

$$\int_{\frac{-\pi}{2}}^{0}\left[\ \frac{d\,\nu}{[\ 1 + e\ \cos(\nu\,)\ ]^2}\ \right]\ =$$

• Hint 3:

$$\frac{A\left(\frac{3\pi}{2}\right)}{A_{total}} = \frac{t\left(\frac{3\pi}{2}\right)}{T}$$

$$\left[\ \frac{1}{1 - e^2}\ \right]\left[\ \frac{2\ \tan^{-1}\left[\ \frac{1\text{-}e}{\sqrt{1 - e^2}}\ \right]}{\sqrt{1 - e^2}}\ -\ e\ \right]$$

N a v

Homework:
Kepler's Second Law (concluded)

Homework …

Hint 4

..... Or Instead of evaluating the BRUTAL Integral ....

$$\int_{\frac{-\pi}{2}}^{0}\left[\ \frac{d\,\nu}{[\ 1 + e\ \cos(\nu\,)\ ]^2}\ \right]\ =$$

$$\left[\ \frac{1}{1 - e^2}\ \right]\left[\ \frac{2\ \tan^{-1}\left[\ \frac{1\text{-}e}{\sqrt{1 - e^2}}\ \right]}{\sqrt{1 - e^2}}\ -\ e\ \right]$$

You can use the area Swept Area or time of flight graphs for Kepler's second law